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Optimal coincidence point results in partially ordered non-Archimedean fuzzy metric spaces

Mujahid Abbas¹, Naeem Saleem² and Manuel De la Sen^{3*}

*Correspondence:

manuel.delasen@ehu.eus

³Institute of Research and Development of Processes, University of the Basque Country, Campus of Leioa (Bizkaia), Barrio Sarriena, P.O. Box 644, Bilbao, 48940, Spain

Full list of author information is available at the end of the article

Abstract

In this paper, we introduce best proximal contractions in complete ordered non-Archimedean fuzzy metric space and obtain some proximal results. The obtained results unify, extend, and generalize some comparable results in the existing literature.

MSC: 47H10; 47H04; 47H07**Keywords:** fuzzy metric space; monotone fuzzy proximal contraction; fuzzy ordered proximal contractions; fuzzy expansive; fuzzy nonexpansive; s -increasing sequence; t -norm

1 Introduction and preliminaries

In 1969, Fan [1], introduced the concept of a best approximation in Hausdorff locally convex topological vector spaces as follows.

Theorem 1.1 *Let X be a nonempty compact convex set in a Hausdorff locally convex topological vector space E and $T : X \rightarrow E$ a continuous mapping, then there exists a fixed point x in X , or there exist a point $x_0 \in X$ and a continuous semi-norm p on E satisfying $\min_{y \in X} p(y - Tx_0) = p(x_0 - T(x_0)) > 0$.*

A fixed point problem is to find a point x in A such that $Tx = x$. There are certain situations where solving an equation $d(x, Tx) = 0$ for x in A is not possible, then a compromise is made on the point x in A where $\inf\{d(y, Tx) : y \in A\}$ is attained, that is, $d(x, Tx) = \inf\{d(y, Tx) : y \in A\}$ holds. Such a point is called an approximate fixed point of T or an approximate solution of an equation $Tx = x$. It is significant to study the conditions that ensure the existence and uniqueness of an approximate fixed point of the mapping T .

Let A and B be two nonempty subsets of X and $T : A \rightarrow B$. Suppose that $d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\}$ is the distance between two sets A and B where $A \cap B = \emptyset$. A point x^* is called a best proximity point of T if $d(x^*, Tx^*) = d(A, B)$. Indeed, if T is a multifunction from A to B then

$$d(x, Tx) \geq d(A, B),$$

for all $x \in A$, always. Note that if $A = B$, then the best proximity point will reduce to a fixed point of the mapping T . Hence the results dealing with the best proximity point problem extend fixed point theory in a natural way.

For more results in this direction, we refer to [2–7] and references therein.

On the other hand, Zadeh [8] introduced the concept of fuzzy sets. Meanwhile Kramosil and Michalek [9] defined fuzzy metric spaces. Later, George and Veeramani [10, 11] further modified the notion of fuzzy metric spaces with the help of a continuous t -norm and generalized the concept of a probabilistic metric space to the fuzzy situation. In this direction, Vetro and Salimi [12] obtained best proximity theorems in non-Archimedean fuzzy metric spaces.

The aim of this paper is to obtain a coincidence best proximity point solution of $M(gx, Tx, t) = M(A, B, t)$ over a nonempty subset A of a partially ordered non-Archimedean fuzzy metric space X , where T is a nonself mapping and g is a self mapping on A . Our results unify, extend, and strengthen various results in [13].

Let us recall some definitions.

Definition 1.2 ([14]) A binary operation $*$: $[0, 1]^2 \rightarrow [0, 1]$ is called a continuous t -norm if

- (1) $*$ is associative, commutative and continuous;
- (2) $a * 1 = a$ for all $a \in [0, 1]$;
- (3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Typical examples of continuous t -norm are \wedge , \cdot , and $*_L$, where, for all $a, b \in [0, 1]$, $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$, and $*_L$ is the Lukasiewicz t -norm defined by $a *_L b = \max\{a + b - 1, 0\}$.

It is easy to check that $*_L \leq \cdot \leq \wedge$. In fact $* \leq \wedge$ for all continuous t -norms $*$.

Definition 1.3 ([11]) Let X be a nonempty set, and $*$ be a continuous t -norm. A fuzzy set M on $X \times X \times [0, +\infty)$ is said to be a fuzzy metric if, for any $x, y, z \in X$, the following conditions hold:

- (i) $M(x, y, t) > 0$,
- (ii) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$,
- (iii) $M(x, y, t) = M(y, x, t)$,
- (iv) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$,
- (v) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous.

The triplet $(X, M, *)$ is called a fuzzy metric space.

Since M is a fuzzy set on $X \times X \times [0, \infty)$, the value $M(x, y, t)$ is regarded as the degree of closeness of x and y with respect to t .

It is well known that for each $x, y \in X$, $M(x, y, \cdot)$ is a nondecreasing function on $(0, +\infty)$ [15].

If we replace (iv) with

- (vi) $M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$,

then the triplet $(X, M, *)$ is said to be a non-Archimedean fuzzy metric space.

As (vi) implies (iv), every non-Archimedean fuzzy metric space is a fuzzy metric space. Also, if we take $s = t$, then (vi) reduces to $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$ for all $t > 0$. And M in this case is said to be a strong fuzzy metric on X .

Each fuzzy metric M on X generates a Hausdorff topology τ_M whose base is the family of open M -balls $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$, where

$$B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}.$$

Note that a sequence $\{x_n\}$ converges to $x \in X$ (with respect to τ_M) if and only if $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$ for all $t > 0$.

Let (X, d) be a metric space. Define $M_d : X \times X \times [0, \infty) \rightarrow [0, 1]$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then (X, M_d, \cdot) is a fuzzy metric space and is called the standard fuzzy metric space induced by a metric d [10]. The topologies τ_{M_d} and τ_d (the topology induced by the metric d) on X are the same. Note that if d is a metric on a set X , then the fuzzy metric space $(X, M_d, *)$ is strong for every continuous t -norm $*$ such that for all $*$ $\leq \cdot$, where M_d is the standard fuzzy metric (see [16]).

A sequence $\{x_n\}$ in a fuzzy metric space X is said to be a Cauchy sequence if for each $t > 0$ and $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$. A fuzzy metric space X is complete [11] if every Cauchy sequence converges in X . A subset A of X is closed if for each convergent sequence $\{x_n\}$ in A with $x_n \rightarrow x$, we have $x \in A$. A subset A of X is compact if each sequence in A has a convergent subsequence.

Lemma 1.4 ([15]) M is a continuous function on $X^2 \times (0, \infty)$.

Definition 1.5 ([7]) Let A and B be two nonempty subsets of a fuzzy metric space $(X, M, *)$. We define $A_0(t)$ and $B_0(t)$ as follows:

$$A_0(t) = \{x \in A : M(x, y, t) = M(A, B, t) \text{ for some } y \in B\},$$

$$B_0(t) = \{y \in B : M(x, y, t) = M(A, B, t) \text{ for some } x \in A\}.$$

The distance of a point $x \in X$ from a nonempty set A for $t > 0$ is defined as

$$M(x, A, t) = \sup_{a \in A} M(x, a, t),$$

and the distance between two nonempty sets A and B for $t > 0$ is defined as

$$M(A, B, t) = \sup\{M(a, b, t) : a \in A, b \in B\}.$$

Definition 1.6 ([4]) Let Ψ be the set of all mappings $\psi : [0, 1] \rightarrow [0, 1]$ satisfying the following properties:

- (i) ψ is continuous and nondecreasing on $(0, 1)$ and $\psi(t) > t$ also $\psi(0) = 0$ and $\psi(1) = 1$.
- (ii) $\lim_{n \rightarrow \infty} \psi^n(t) = 1$ if and only if $t = 1$.

Let Λ be the set of all mappings $\eta : [0, 1] \rightarrow [0, 1]$ which satisfy the following properties:

- (i) η is continuous and strictly decreasing on $(0, 1)$ and $\eta(t) < t$ for all $t \in (0, 1)$,
- (ii) $\eta(1) = 1$ and $\eta(0) = 0$.

If we take $\eta(t) = 2t - t^2$, then $\eta \in \Lambda$ and hence $\Lambda \neq \emptyset$.

2 Best proximity point in partially ordered non-Archimedean fuzzy metric space

Definition 2.1 Let A be a nonempty subset of a non-Archimedean fuzzy metric space $(X, M, *)$. A self mapping f on A is said to be (a) fuzzy isometry if $M(fx, fy, t) = M(x, y, t)$ for all $x, y \in A$ and $t > 0$ (b) fuzzy expansive if, for any $x, y \in A$ and $t > 0$, we have $M(fx, fy, t) \leq M(x, y, t)$, (c) fuzzy nonexpansive if, for any $x, y \in A$ and $t > 0$, we have $M(fx, fy, t) \geq M(x, y, t)$.

Example 2.2 Let $X = [0, 1] \times \mathbb{R}$ and $d : X \times X \rightarrow \mathbb{R}$ be a usual metric on X . Let $A = \{(0, x) : x \in \mathbb{R}\}$. Note that (X, M_d, \cdot) is non-Archimedean fuzzy metric space, where M_d is standard fuzzy metric induced by d . Define the mapping $f : A \rightarrow A$ by $f(0, x) = (0, -x)$. Note that $M_d(w, u, t) = \frac{t}{t+|x-y|} = M(fw, fu, t)$, where $w = (0, x)$, $u = (0, y) \in A$.

Note that every fuzzy isometry is fuzzy expansive but the converse does not hold in general.

Example 2.3 Let $X = [0, 4] \times \mathbb{R}$ and $d : X \times X \rightarrow \mathbb{R}$ be a usual metric on X . Let $A = \{(0, x) : x \in \mathbb{R}\}$. Note that (X, M_d, \cdot) is a non-Archimedean fuzzy metric space, where M_d is the standard fuzzy metric induced by d . Define the mapping $f : A \rightarrow A$ by

$$f(0, x) = 100(0, x).$$

If $x = (0, 0)$ and $y = (0, 4)$ then $M(x, y, t) = \frac{t}{t+4}$ and $M(fx, fy, t) = \frac{t}{t+400}$. This shows that f is fuzzy expansive but not a fuzzy isometry.

Example 2.4 Let $X = [0, 1] \times \mathbb{R}$, $d : X \times X \rightarrow \mathbb{R}$ a usual metric on X and $A = \{(0, x) : x \in \mathbb{R}\}$. Define a mapping $f : A \rightarrow A$ by

$$f(0, x) = \left(0, \frac{x}{10}\right).$$

If $x = (0, 0)$ and $y = (0, 1)$ then $M(x, y, t) = \frac{t}{t+1}$ and $M(fx, fy, t) = \frac{t}{t+\frac{1}{10}} \geq \frac{t}{t+1} = M(x, y, t)$. Thus f is fuzzy nonexpansive but not a fuzzy isometry.

Note that the fuzzy expansive and nonexpansive mapping are fuzzy isometries. However, the converse is not true in general.

Definition 2.5 Let A, B be nonempty subsets of a non-Archimedean fuzzy metric space $(X, M, *)$. A set B is said to be fuzzy approximatively compact with respect to A if for every sequence $\{y_n\}$ in B and for some $x \in A$, $M(x, y_n, t) \rightarrow M(x, B, t)$ implies that $x \in A_0(t)$.

Definition 2.6 ([17]) A sequence $\{t_n\}$ of positive real numbers is said to be s -increasing if there exists $n_0 \in \mathbb{N}$ such that $t_{n+1} \geq t_n + 1$ for all $n \geq n_0$.

Definition 2.7 (compare [18]) A fuzzy metric space $(X, M, *)$ is said to satisfy *property T* if, for any s -increasing sequence, there exists $n_0 \in \mathbb{N}$ such that $\prod_{n \geq n_0}^{\infty} M(x, y, t_n) \geq 1 - \varepsilon$ for all $n \geq n_0$.

A 4-tuple $(X, M, *, \preceq)$ is called a partially ordered fuzzy metric space if (X, \preceq) is a partially ordered set and $(X, M, *)$ is a non-Archimedean fuzzy metric space. Unless otherwise stated, it is assumed that A, B are nonempty closed subsets of partially ordered fuzzy metric space $(X, M, *, \preceq)$.

Definition 2.8 ([13]) A mapping $T : A \rightarrow B$ is called (a) nondecreasing or order preserving if, for any x, y in A with $x \preceq y$, we have $Tx \preceq Ty$; (b) an ordered reversing if, for any x, y in A with $x \preceq y$, we have $Tx \succeq Ty$; (c) monotone if it is order preserving or order reversing.

Definition 2.9 ([19]) Let A, B be nonempty subsets of partially ordered fuzzy metric space $(X, M, *, \preceq)$ and $\psi : [0, 1] \rightarrow [0, 1]$ be a continuous mapping. A mapping $T : A \rightarrow B$ is said to be a fuzzy ordered ψ -contraction if, for any $x, y \in A$ with $x \preceq y$, we have $M(Tx, Ty, t) \geq \psi[M(x, y, t)]$ for all $t > 0$.

Definition 2.10 A mapping $T : A \rightarrow B$ is called a fuzzy ordered proximal ψ -contraction of type-I if, for any u, v, x , and y in A , the following condition holds:

$$\left. \begin{array}{l} x \preceq y \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array} \right\} \implies M(u, v, t) \geq \psi[M(x, y, t)], \quad \text{where } \psi \in \Psi.$$

Definition 2.11 A mapping $T : A \rightarrow B$ is said to be a fuzzy ordered proximal ψ -contraction of type-II if, for any u, v, x , and y in A , and for some $\alpha \in (0, 1)$, the following condition holds:

$$\left. \begin{array}{l} x \preceq y \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array} \right\} \implies M(u, v, t) \geq \psi \left[M \left(x, y, \frac{t}{\alpha} \right) \right], \quad \text{where } \psi \in \Psi.$$

Definition 2.12 A mapping $T : A \rightarrow B$ is called a fuzzy ordered η -proximal contraction if, for any u, v, x , and y in A , the following condition holds:

$$\left. \begin{array}{l} x \preceq y \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array} \right\} \implies M(x, y, t) \leq \eta[M(u, v, t)], \quad \text{where } \eta \in \Lambda.$$

Definition 2.13 A mapping $T : A \rightarrow B$ is said to be a proximal fuzzy order preserving if, for any u, v, x , and y in A , the following implication holds:

$$\left. \begin{array}{l} x \preceq y \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array} \right\} \implies u \preceq v.$$

If $A = B$, then a proximal fuzzy order preserving mapping will become fuzzy order preserving.

Definition 2.14 A mapping $T : A \rightarrow B$ is said to be a proximal fuzzy order reversing if for any u, v, x , and y in A , the following implication holds:

$$\left. \begin{array}{l} x \preceq y \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array} \right\} \implies u \succeq v.$$

If $A = B$, then proximal fuzzy order reversing mapping will become fuzzy order reversing.

Definition 2.15 A point x in A is said to be an optimal coincidence point of the pair of mappings (g, T) , where $T : A \rightarrow B$ is a nonself mapping and $g : A \rightarrow A$ is a self mapping if

$$M(gx, Tx, t) = M(A, B, t)$$

holds.

From now on, we use the notation $\Delta_{(t)}$ for a set $\{(x, y) \in A_0(t) \times A_0(t) : \text{either } x \preceq y \text{ or } y \preceq x\}$.

We start with the following result.

Theorem 2.16 Let $T : A \rightarrow B$ be continuous, proximally monotone, and proximal fuzzy ordered ψ -contraction of type-I, $g : A \rightarrow A$ surjective, fuzzy expansive and inverse monotone mapping. Suppose that each pair of elements in X has a lower and upper bound and for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty such that $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$. If there exist some elements x_0 and x_1 in $A_0(t)$ such that

$$M(gx_1, Tx_0, t) = M(A, B, t) \quad \text{with } (x_0, x_1) \in \Delta_{(t)},$$

then there exists a unique element $x^* \in A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$, that is, x^* is an optimal coincidence point of the pair (g, T) . Further, for any fixed element $\bar{x}_0 \in A_0(t)$, the sequence $\{\bar{x}_n\}$ defined by $M(g\bar{x}_{n+1}, T\bar{x}_n, t) = M(A, B, t)$ converges to x^* .

Proof Let x_0 and x_1 be given points in $A_0(t)$ such that

$$M(gx_1, Tx_0, t) = M(A, B, t) \quad \text{with } (x_0, x_1) \in \Delta_{(t)}. \quad (1)$$

Since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$, and $A_0(t) \subseteq g(A_0(t))$, we can choose an element $x_2 \in A_0(t)$ such that

$$M(gx_2, Tx_1, t) = M(A, B, t). \quad (2)$$

As T is proximally monotone, we have $(gx_1, gx_2) \in \Delta_{(t)}$ which further implies that $(x_1, x_2) \in \Delta_{(t)}$. Continuing this way, we obtain a sequence $\{x_n\}$ in $A_0(t)$, such that it satisfies

$$M(gx_n, Tx_{n-1}, t) = M(A, B, t) \quad \text{with } (x_{n-1}, x_n) \in \Delta_{(t)} \quad (3)$$

for each positive integer n . Having chosen x_n , one can find a point x_{n+1} in $A_0(t)$ such that

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t). \quad (4)$$

Since $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, T is proximally monotone mapping, so from (3) and (4) it follows that $(gx_n, gx_{n+1}) \in \Delta_{(t)}$ and $(x_n, x_{n+1}) \in \Delta_{(t)}$. Note that

$$M(x_n, x_{n+1}, t) \geq M(gx_n, gx_{n+1}, t) \geq \psi[M(x_{n-1}, x_n, t)]. \quad (5)$$

Denote $M(x_n, x_{n+1}, t) = \tau_n(t)$ for all $t > 0$, $n \in \mathbb{N} \cup \{0\}$. The above inequality becomes

$$\tau_n(t) \geq \psi(\tau_{n-1}(t)) > \tau_{n-1}(t) \quad (6)$$

and

$$\tau_n(t) > \tau_{n-1}(t).$$

Thus $\{\tau_n(t)\}$ is an increasing sequence for all $t > 0$. Consequently, there exists $\tau(t) \leq 1$ such that $\lim_{n \rightarrow +\infty} \tau_n(t) = \tau(t)$. Note that $\tau(t) = 1$. If not, there exists some $t_0 > 0$ such that $\tau(t_0) < 1$. Also, $\tau_n(t_0) \leq \tau(t_0)$. By taking limit as $n \rightarrow \infty$ on both sides of (6), we have

$$\tau(t_0) \geq \psi(\tau(t_0)) > \tau(t_0),$$

a contradiction. Hence $\tau(t) = 1$. Now we show that $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon \in (0, 1)$ and $t_0 > 0$ such that for all $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$, with $m_k > n_k \geq k$ such that

$$M(x_{m_k}, x_{n_k}, t_0) \leq 1 - \varepsilon. \quad (7)$$

Assume that m_k is the least integer exceeding n_k and satisfying the above inequality, then we have

$$M(x_{m_k-1}, x_{n_k}, t_0) > 1 - \varepsilon. \quad (8)$$

So, for all k ,

$$\begin{aligned} 1 - \varepsilon &\geq M(x_{m_k}, x_{n_k}, t_0) \\ &\geq M(x_{m_k}, x_{m_k-1}, t_0) * M(x_{m_k-1}, x_{n_k}, t_0) \\ &> \tau_{m_k}(t_0) * (1 - \varepsilon). \end{aligned} \quad (9)$$

On taking the limit as $k \rightarrow \infty$ on both sides of the above inequality, we obtain $\lim_{k \rightarrow +\infty} M(x_{m_k}, x_{n_k}, t_0) = 1 - \varepsilon$. Note that

$$M(x_{m_k+1}, x_{n_k+1}, t_0) \geq M(x_{m_k+1}, x_{m_k}, t_0) * M(x_{m_k}, x_{n_k}, t_0) * M(x_{n_k}, x_{n_k+1}, t_0)$$

and

$$M(x_{m_k}, x_{n_k}, t_0) \geq M(x_{m_k}, x_{m_k+1}, t_0) * M(x_{m_k+1}, x_{n_k+1}, t_0) * M(x_{n_k+1}, x_{n_k}, t_0),$$

imply that

$$\lim_{k \rightarrow +\infty} M(x_{m_k+1}, x_{n_k+1}, t_0) = 1 - \varepsilon.$$

From (4), we have

$$M(gx_{m_k+1}, Tx_{m_k}, t_0) = M(A, B, t_0) \quad \text{and} \quad M(gx_{n_k+1}, Tx_{n_k}, t_0) = M(A, B, t_0).$$

Thus

$$M(x_{m_k+1}, x_{n_k+1}, t_0) \geq M(gx_{m_k+1}, gx_{n_k+1}, t_0) \geq \psi[M(x_{m_k}, x_{n_k}, t_0)].$$

On taking the limit as $k \rightarrow \infty$ in the above inequality, we get $1 - \varepsilon \geq \psi(1 - \varepsilon) > 1 - \varepsilon$, a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in the closed subset $A(t)$ of complete partially ordered fuzzy metric space $(X, M, *, \leq)$. There exists $x^* \in A(t)$ such that $\lim_{n \rightarrow \infty} M(x_n, x^*, t) = 1$, for all $t > 0$. This further implies that

$$M(gx^*, Tx^*, t) = \lim_{n \rightarrow \infty} M(gx_{n+1}, Tx_n, t) = M(A, B, t).$$

Hence $x^* \in A_0(t)$ is the optimal coincidence point of a pair $\{g, T\}$. To prove the uniqueness of x^* ; We show that, for any fixed element $\bar{x}_0 \in A_0(t)$, the sequence $\{\bar{x}_n\} \in A_0(t)$ defined by $M(g\bar{x}_{n+1}, T\bar{x}_n, t) = M(A, B, t)$ converges to x^* . Suppose that there is another element $\bar{x}_0 \in A(t)$ such that $0 < M(x_0, \bar{x}_0, t) < 1$ for all $t > 0$ satisfying

$$M(g\bar{x}_0, T\bar{x}_0, t) = M(A, B, t). \quad (10)$$

Suppose that $(\bar{x}_0, x_0) \in \Delta_{(t)}$, that is, $\bar{x}_0 \leq x_0$ or $\bar{x}_0 \geq x_0$. Then by the given assumption, we have

$$M(\bar{x}_0, x_0, t) \geq M(g\bar{x}_0, gx_0, t) \geq \psi(M(\bar{x}_0, x_0, t)) > M(\bar{x}_0, x_0, t)$$

a contradiction. So x^* is unique. If $(\bar{x}_0, x_0) \notin \Delta_{(t)}$, then by assumption, suppose that u_0 be a lower bound of x_0 and \bar{x}_0 , also assume that \bar{u}_0 is an upper bound of x_0 and \bar{x}_0 . That is,

$$\bar{u}_0 \geq x_0 \geq u_0 \quad \text{or} \quad \bar{u}_0 \geq \bar{x}_0 \geq u_0.$$

Recursively, construct the sequences $\{u_n\}$ and $\{\bar{u}_n\}$, such that

$$M(gu_{n+1}, Tu_n, t) = M(A, B, t) \quad \text{and} \quad M(g\bar{u}_{n+1}, T\bar{u}_n, t) = M(A, B, t).$$

The proximal monotonicity of the mapping T and the monotonicity of the inverse of g imply that

$$\bar{u}_n \geq \bar{x}_n \geq u_n \quad \text{or} \quad \bar{u}_n \leq \bar{x}_n \leq u_n.$$

Since $(x_0, u_0) \in \Delta_{(t)}$, also $(x_0, \bar{u}_0) \in \Delta_{(t)}$, similarly we have $(x_n, u_n) \in \Delta_{(t)}$ and $(x_n, \bar{u}_n) \in \Delta_{(t)}$, therefore

$$\lim_{n \rightarrow \infty} \bar{u}_n = \lim_{n \rightarrow \infty} u_n = x^*.$$

Hence

$$\lim_{n \rightarrow \infty} \bar{x}_n = x^*.$$

This completes the proof. \square

Example 2.17 Let $X = [0, 1] \times \mathbb{R}$ and \leq be the usual order on \mathbb{R}^2 , that is, $(x, y) \leq (z, w)$ if and only if $x \leq z$ and $y \leq w$. Suppose that $A = \{(-1, x) : \text{for all } x \in \mathbb{R}\}$ and $B = \{(1, y) : \text{for all } y \in \mathbb{R}\}$. $(X, M, *, \leq)$ is a complete ordered metric space under $M(x, y, t) = \frac{t}{t+d(x, y)}$ for all $t > 0$, where $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ for all $x = (x_1, y_1)$, $y = (x_2, y_2)$. Note that $M(A, B, t) = \frac{t}{t+2}$, $A_0(t) = A$, and $B_0(t) = B$. Define $T : A \rightarrow B$ by

$$T(-1, x) = \left(1, \frac{x}{2}\right).$$

Let $g : A \rightarrow A$ be defined by $g(-1, x) = (-1, 2x)$. Note that g is fuzzy expansive and its inverse is monotone. Obviously, $T(A_0(t)) = B_0(t)$, and $A_0(t) = g(A_0(t))$. Note that $u = (-1, \frac{y_1}{4})$, $v = (-1, \frac{y_2}{4})$, $x = (-1, y_1)$, and $y = (-1, y_2) \in A$ satisfy

$$M(gu, Tx, t) = M(A, B, t), \quad (11)$$

$$M(gv, Ty, t) = M(A, B, t). \quad (12)$$

Also, note that

$$M(gu, gv, t) = M\left(\left(-1, \frac{y_1}{2}\right), \left(-1, \frac{y_2}{2}\right), t\right) \geq \psi(M((-1, y_1), (-1, y_2), t)) = \psi(M(x, y, t)),$$

where $\psi(t) = \sqrt{t}$. Thus all conditions of Theorem 2.16 are satisfied. However, $(-1, 0)$ is the optimal coincidence point of g and T , satisfying the conclusion of the theorem.

The above example shows that our result is a potential generalization of Theorem 3.1 in [13].

Corollary 2.18 Let $T : A \rightarrow B$ is continuous, proximally monotone, and proximal fuzzy ordered ψ -contraction of type-I, $g : A \rightarrow A$ surjective, a fuzzy isometry, and an inverse monotone mapping. Suppose that each pair of elements in X has a lower and upper bound, for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty such that $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$. If there exist some elements x_0 and x_1 in $A_0(t)$ such that

$$M(gx_1, Tx_0, t) = M(A, B, t) \quad \text{with } (x_0, x_1) \in \Delta_{(t)},$$

then there exists a unique element $x^* \in A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$. Further, for any fixed element $\bar{x}_0 \in A_0(t)$, the sequence $\{\bar{x}_n\}$ defined by $M(g\bar{x}_{n+1}, T\bar{x}_n, t) = M(A, B, t)$ converges to x^* .

Proof Every fuzzy isometry is fuzzy expansive, and this corollary satisfies all the conditions of Theorem 2.16. \square

Example 2.19 Let $X = [-1, 1] \times \mathbb{R}$ and \preceq a usual order on \mathbb{R}^2 . Let $A = \{(-1, x) : \text{for all } x \in \mathbb{R}\}$, $B = \{(1, y) : \text{for all } y \in \mathbb{R}\}$, and $(X, M, *, \preceq)$ a complete fuzzy ordered metric space as given in Example 2.17. Note that $M(A, B, t) = \frac{t}{t+2}$, $A_0(t) = A$ and $B_0(t) = B$. Define $T : A \rightarrow B$ by

$$T(-1, x) = \left(1, \frac{x}{5}\right).$$

Let $g : A \rightarrow A$ be defined by $g(-1, x) = (-1, -x)$. Note that g is a fuzzy isometry and its inverse is monotone. Obviously, $T(A_0(t)) = B_0(t)$, and $A_0(t) = g(A_0(t))$. Note that $u = (-1, -\frac{y_1}{5})$, $v = (-1, -\frac{y_2}{5})$, $x = (-1, y_1)$, and $y = (-1, y_2) \in A_0(t)$ satisfy

$$M(gu, Tx, t) = M(A, B, t),$$

$$M(gv, Ty, t) = M(A, B, t).$$

Also, note that

$$M(gu, gv, t) = M\left(\left(-1, \frac{y_1}{5}\right), \left(-1, \frac{y_2}{5}\right), t\right) \geq \psi\left(M((-1, y_1), (-1, y_2), t)\right) = \psi(M(A, B, t)),$$

where $\psi(t) = \sqrt{t}$. All conditions of Corollary 2.18 are satisfied. Moreover, $(-1, 0)$ is an optimal coincidence point of g and T .

Corollary 2.20 Let $T : A \rightarrow B$ be a continuous, proximally monotone, and proximal fuzzy ordered ψ -contraction of type-I. Suppose that each pair of elements in X has a lower and upper bound for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty such that $T(A_0(t)) \subseteq B_0(t)$. If there exist some elements x_0 and x_1 in $A_0(t)$ such that

$$M(x_1, Tx_0, t) = M(A, B, t) \quad \text{with } (x_0, x_1) \in \Delta_{(t)},$$

then there exists a unique element $x^* \in A_0(t)$ such that $M(x^*, Tx^*, t) = M(A, B, t)$. Further, for any fixed element $\bar{x}_0 \in A_0(t)$, the sequence $\{\bar{x}_n\}$ defined by $M(\bar{x}_{n+1}, T\bar{x}_n, t) = M(A, B, t)$ converges to x^* .

Proof This corollary satisfies all the conditions of Theorem 2.16 by taking $gx = I_A$ (an identity mapping on A). \square

3 Best proximity point in partially ordered non-Archimedean fuzzy metric spaces for proximal ψ -contractions of type-II

Theorem 3.1 Let $T : A \rightarrow B$ be continuous, proximally monotone, and proximal ordered fuzzy ψ -contraction of type-II, $g : A \rightarrow A$ surjective, fuzzy expansive, and inverse monotone mapping. Suppose that each pair of elements in X has a lower and upper bound, and an s -increasing sequence $\{t_n\}$ satisfying property T, for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty

such that $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$. If there exist some elements x_0 and x_1 in $A_0(t)$ such that

$$M(gx_1, Tx_0, t) = M(A, B, t) \quad \text{with } (x_0, x_1) \in \Delta_{(t)},$$

then there exists a unique element $x \in A_0(t)$ such that $M(gx, Tx, t) = M(A, B, t)$. Further, for any fixed element $\bar{x}_0 \in A_0(t)$, the sequence $\{\bar{x}_n\} \in A_0(t)$, defined by $M(g\bar{x}_{n+1}, T\bar{x}_n, t) = M(A, B, t)$, converges to x .

Proof Let x_0 and x_1 be given elements in $A_0(t)$. such that

$$M(gx_1, Tx_0, t) = M(A, B, t) \quad \text{with } (x_0, x_1) \in \Delta_{(t)}. \quad (13)$$

Since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$, $A_0(t) \subseteq T(A_0(t)) \subseteq B_0(t)$, and $A_0(t) \subseteq g(A_0(t))$, it follows that there exists an element $x_2 \in A_0(t)$ such that it satisfies

$$M(gx_2, Tx_1, t) = M(A, B, t). \quad (14)$$

As T is proximal monotone, we have $(gx_1, gx_2) \in \Delta_{(t)}$, which further implies that $(x_1, x_2) \in \Delta_{(t)}$. Continuing this way, we obtain a sequence $\{x_n\}$ in $A_0(t)$ such that

$$M(gx_n, Tx_{n-1}, t) = M(A, B, t) \quad \text{with } (x_{n-1}, x_n) \in \Delta_{(t)} \quad (15)$$

for each positive integer n . Hence after finding x_n , we can find an element x_{n+1} in $A_0(t)$ such that

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t). \quad (16)$$

Since $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, T is proximally monotone mapping, so from (15) and (16), it follows that $(gx_n, gx_{n+1}) \in \Delta_{(t)}$ and $(x_n, x_{n+1}) \in \Delta_{(t)}$. Note that

$$M(x_{n+1}, x_n, t) \geq M(gx_{n+1}, gx_n, t) \geq \psi \left(M \left(x_n, x_{n-1}, \frac{t}{\alpha} \right) \right). \quad (17)$$

for all $n \geq 0$. Recursively,

$$\begin{aligned} M(x_{n+1}, x_n, t) &\geq \psi \left(M \left(x_{n+1}, x_n, \frac{t}{\alpha} \right) \right) \geq \psi^2 \left(M \left(x_n, x_{n-1}, \frac{t}{\alpha^2} \right) \right) \geq \cdots \\ &\geq \psi^n \left(M \left(x_1, x_0, \frac{t}{\alpha^n} \right) \right) > M \left(x_1, x_0, \frac{t}{\alpha^n} \right), \end{aligned} \quad (18)$$

for all $t > 0$ and $m, n \in \mathbb{N}$, where $m \geq n$, so we have

$$\begin{aligned} M(x_n, x_m, t) &\geq M(x_n, x_{n+1}, t) * M(x_{n+1}, x_{n+2}, t) * M(x_{n+2}, x_{n+3}, t) * \cdots * M(x_{m-1}, x_m, t) \\ &> M \left(x_0, x_1, \frac{t}{\alpha^n} \right) * M \left(x_0, x_1, \frac{t}{\alpha^{n+1}} \right) * \cdots * M \left(x_0, x_1, \frac{t}{\alpha^{m-1}} \right) \\ &> \prod_{i=n}^{\infty} M \left(x_0, x_1, \frac{t}{\alpha^i} \right), \end{aligned}$$

where $t_i = \frac{t}{\alpha^i}$. As $\lim_{n \rightarrow \infty} (t_{n+1} - t_n) = \infty$, $\{t_n\}$ is an s -increasing sequence satisfying the property T . Consequently for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, so we have $\prod_{n=1}^{\infty} M(x_0, x_1, t_n) \geq 1 - \varepsilon$ for all $n \geq n_0$. Hence we obtain $M(x_n, x_m, t) \geq 1 - \varepsilon$ for all $n, m \geq n_0$ and $\{x_n\}$ is a Cauchy sequence in $A(t)$. By the completeness of X , there exists x in $A(t)$ such that $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$ for all $t > 0$. This further implies that

$$\begin{aligned} M(gx, B, t) &\geq M(gx, Tx_n, t) \\ &\geq M(gx, gx_{n+1}, t) * M(gx_{n+1}, Tx_n, t) \\ &= M(gx, gx_{n+1}, t) * M(A, B, t) \\ &\geq M(gx, gx_{n+1}, t) * M(gx, B, t). \end{aligned}$$

Since g is continuous, the sequence $\{gx_n\}$ converges to gx . Therefore, $M(gx, Tx_n, t) \rightarrow M(gx, B, t)$. Since $B(t)$ is fuzzy approximately compact with respect to $A(t)$, $\{Tx_n\}$ has a subsequence which converges to y in $B(t)$ such that

$$M(gx, y, t) = M(A, B, t),$$

for some $y \in B(t)$, hence $gx \in A_0(t)$ implies $gx = gu$ for some $u \in A_0(t)$. Hence $M(x, u, t) \geq M(gx, gu, t) = 1$, which implies that $M(x, u, t) = 1$. Thus x and u are identical, and hence $x \in A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$,

$$M(z, Tx, t) = M(A, B, t) \quad (19)$$

for some z in $A(t)$. From (16) and (19) we obtain

$$M(gx_{n+1}, z, t) \geq \psi \left(M \left(x, x_n, \frac{t}{\alpha} \right) \right). \quad (20)$$

Taking the limit as $n \rightarrow \infty$, the above inequality becomes

$$\lim_{n \rightarrow \infty} M(gx_{n+1}, z, t) \geq \lim_{n \rightarrow \infty} \psi \left(M \left(x, x_n, \frac{t}{\alpha} \right) \right) = 1,$$

which shows that $\{gx_n\}$ converges to z

$$M(gx_n, z, t) = 1. \quad (21)$$

Since g is continuous, the sequence $\{gx_n\}$ converges to gx such that

$$M(gx_n, gx, t) = 1. \quad (22)$$

Hence we have $gx = z$,

$$M(gx, Tx, t) = M(A, B, t) = M(z, Tx, t). \quad (23)$$

Suppose that there is another element x^* such that

$$M(gx^*, Tx^*, t) = M(A, B, t). \quad (24)$$

First suppose that $(x, x^*) \in \Delta_{(t)}$. From (23) and (24), it follows that

$$M(x, x^*, t) \geq M(gx, gx^*, t) \geq \psi\left(M\left(x, x^*, \frac{t}{\alpha}\right)\right),$$

which further implies that

$$M(x, x^*, t) > M\left(x, x^*, \frac{t}{\alpha}\right),$$

a contradiction. Hence x is unique.

Now, suppose that $(x, x^*) \notin \Delta_{(t)}$. Let \bar{x}_0 be any element in $A_0(t)$, u_0 and \bar{u}_0 be lower and upper bounds of x_0 and \bar{x}_0 , respectively such that

$$\bar{u}_0 \geq x_0 \geq u_0 \quad \text{or} \quad \bar{u}_0 \geq \bar{x}_0 \geq u_0.$$

Recursively, we can find sequences $\{u_n\}$ and $\{\bar{u}_n\}$ such that

$$M(gu_{n+1}, Tu_n, t) = M(A, B, t) \quad \text{and} \quad M(g\bar{u}_{n+1}, T\bar{u}_n, t) = M(A, B, t).$$

The proximal monotonicity of the mapping T and the monotonicity of the inverse of g implies that

$$\bar{u}_n \geq \bar{x}_n \geq u_n \quad \text{or} \quad \bar{u}_n \leq \bar{x}_n \leq u_n.$$

Since $(x_0, u_0) \in \Delta_{(t)}$, also $(x_0, \bar{u}_0) \in \Delta_{(t)}$. It follows that

$$\lim_{n \rightarrow \infty} \bar{u}_n = \lim_{n \rightarrow \infty} u_n = x^*.$$

Hence

$$\lim_{n \rightarrow \infty} x_n = x^*.$$

This completes the proof. \square

Example 3.2 Let $X = [0, 2] \times \mathbb{R}$ and \leq a usual order on \mathbb{R}^2 . Let $A = \{(0, x) : x \geq 0 \text{ and } x \in \mathbb{R}\}$, $B = \{(2, y) : \text{for all } y \in \mathbb{R}\}$, and $(X, M, *, \leq)$ a complete fuzzy ordered metric space as given in Example 2.17. Note that $A_0(t) = A$, $B_0(t) = \{(2, y) : y \geq 0 \text{ and } y \in \mathbb{R}\}$. Define $T : A \rightarrow B$ by

$$T(0, x) = \left(2, \frac{x}{10}\right).$$

Let $g : A \rightarrow A$ be defined by $g(0, x) = (0, 10x)$. Note that g is a fuzzy expansive and its inverse is monotone. Obviously, $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$. Note that $u = (0, \frac{y_1}{100})$, $v = (0, \frac{y_2}{100})$, $x = (0, y_1)$, and $y = (0, y_2) \in A_0(t)$ satisfy

$$M(gu, Tx, t) = M(A, B, t),$$

$$M(gv, Ty, t) = M(A, B, t).$$

Also, note that

$$\begin{aligned} M(gu, gv, t) &= M\left(\left(0, \frac{y_1}{10}\right), \left(0, \frac{y_2}{10}\right), t\right) \geq \psi\left(M\left((0, y_1), (0, y_2), \frac{t}{\alpha}\right)\right) \\ &= \psi\left(M\left(x, y, \frac{t}{\alpha}\right)\right), \end{aligned}$$

where $\psi(t) = \sqrt{t}$ and for all $\alpha \in [\frac{1}{10}, 1]$. All conditions of Theorem 3.1 are satisfied. Moreover, $(0, 0)$ is optimal coincidence point of g and T .

Corollary 3.3 *Let $T : A \rightarrow B$ be continuous, proximally monotone, and proximal ordered fuzzy ψ -contraction of type-II, $g : A \rightarrow A$ surjective, fuzzy isometry and inverse monotone mapping. Suppose that each pair of elements in X has a lower and upper bound, and an s -increasing sequence $\{t_n\}$ satisfying property T , for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty such that $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$. If there exist some elements x_0 and x_1 in $A_0(t)$ such that*

$$M(gx_1, Tx_0, t) = M(A, B, t) \quad \text{and} \quad (x_0, x_1) \in \Delta_{(t)},$$

then there exists a unique element $x^ \in A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$. Further, for any fixed element $\bar{x}_0 \in A_0(t)$, the sequence $\{\bar{x}_n\} \in A_0(t)$, defined by $M(g\bar{x}_{n+1}, T\bar{x}_n, t) = M(A, B, t)$, converges to x^* .*

Proof Here the T satisfy all the conditions of Theorem 3.1 if we consider g as fuzzy isometry mapping. \square

Corollary 3.4 *Let $T : A \rightarrow B$ be continuous, proximally monotone, and proximal ordered fuzzy ψ -contraction of type-II. Suppose that each pair of elements in X has a lower and upper bound, and an s -increasing sequence $\{t_n\}$ satisfying property T , for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty such that $T(A_0(t)) \subseteq B_0(t)$.*

Then there exists a unique element $x^ \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$. Further, for any fixed element $x_0 \in A_0(t)$, the sequence $\{x_n\} \in A_0(t)$, defined by $M(x_{n+1}, Tx_n, t) = M(A, B, t)$, converges to x^* .*

Proof Here the T satisfy all the conditions of Theorem 3.1 if $g(x) = I_A$ (an identity mapping on A). \square

4 Best proximity point in partially ordered non-Archimedean fuzzy metric spaces for proximal η -contractions

Theorem 4.1 *Let $T : A \rightarrow B$ be continuous, proximally monotone, and proximal fuzzy ordered η -contraction such that, for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty with $T(A_0(t)) \subseteq B_0(t)$, $g : A \rightarrow A$ surjective, fuzzy nonexpansive and inverse monotone mapping with $A_0(t) \subseteq g(A_0(t))$ for any $t > 0$. If there exist some elements x_0 and x_1 in $A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$ with $(x_0, x_1) \in \Delta_{(t)}$, then there exists a unique element $x^* \in A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$ provided that each pair of elements in X has a lower and upper bound. Further, for any fixed element $\bar{x}_0 \in A_0(t)$, the sequence $\{\bar{x}_n\}$ defined by $M(g\bar{x}_{n+1}, T\bar{x}_n, t) = M(A, B, t)$ converges to x^* .*

Proof Let x_0 and x_1 be given points in $A_0(t)$ such that

$$M(gx_1, Tx_0, t) = M(A, B, t) \quad \text{with } (x_0, x_1) \in \Delta_{(t)}. \quad (25)$$

Since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can choose an element $x_2 \in A_0(t)$ such that

$$M(gx_2, Tx_1, t) = M(A, B, t). \quad (26)$$

As T is proximally monotone, we have $(gx_1, gx_2) \in \Delta_{(t)}$, which further implies that $(x_1, x_2) \in \Delta_{(t)}$. Continuing this way, we can obtain a sequence $\{x_n\}$ in $A_0(t)$, such that it satisfies

$$M(gx_n, Tx_{n-1}, t) = M(A, B, t) \quad \text{with } (x_{n-1}, x_n) \in \Delta_{(t)}. \quad (27)$$

for each positive integer n . Having chosen x_n , one can find a point x_{n+1} in $A_0(t)$ such that

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t). \quad (28)$$

Since $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, T is proximally monotone mapping, so from (27) and (28) it follows that $(gx_n, gx_{n+1}) \in \Delta_{(t)}$ and $(x_n, x_{n+1}) \in \Delta_{(t)}$. Note that

$$M(x_n, x_{n-1}, t) \leq \eta[M(gx_{n+1}, gx_n, t)] \leq \eta[M(x_{n+1}, x_n, t)] < M(x_{n+1}, x_n, t). \quad (29)$$

Denote $M(x_n, x_{n+1}, t) = \tau_n(t)$ for all $t > 0$, $n \in \mathbb{N} \cup \{0\}$. The above inequality becomes

$$\tau_{n-1}(t) \leq \eta(\tau_n(t)) < \tau_n(t). \quad (30)$$

Thus $\{\tau_n(t)\}$ is an increasing sequence for each $t > 0$. Consequently, $\lim_{n \rightarrow +\infty} \tau_n(t) = \tau(t)$. We claim that $\tau(t) = 1$ for each $t > 0$. If not, there exist some $t_0 > 0$ such that $\tau(t_0) < 1$. Also, $\tau_n(t_0) \leq \tau(t_0)$. On taking limit as $n \rightarrow \infty$ on both sides of (30), we have $\tau(t_0) \leq \eta(\tau(t_0)) < \tau(t_0)$, a contradiction. Hence $\tau(t) = 1$ for each $t > 0$. Now we show that $\{x_n\}$ is a Cauchy sequence. If not, then there exist some $\varepsilon \in (0, 1)$ and $t_0 > 0$ such that for all $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$, with $m_k > n_k \geq k$ such that

$$M(x_{m_k}, x_{n_k}, t_0) \leq 1 - \varepsilon. \quad (31)$$

If m_k is the least integer exceeding n_k and satisfying the above inequality, then

$$M(x_{m_k-1}, x_{n_k}, t_0) > 1 - \varepsilon. \quad (32)$$

So, for all k ,

$$\begin{aligned} 1 - \varepsilon &\geq M(x_{m_k}, x_{n_k}, t_0) \\ &\geq M(x_{m_k}, x_{m_k-1}, t_0) * M(x_{m_k-1}, x_{n_k}, t_0) \\ &> \tau_{m_k}(t_0) * (1 - \varepsilon). \end{aligned} \quad (33)$$

On taking the limit as $k \rightarrow \infty$ on both sides of above inequality, we obtain $\lim_{k \rightarrow +\infty} M(x_{m_k}, x_{n_k}, t_0) = 1 - \varepsilon$. Note that

$$M(x_{m_k+1}, x_{n_k+1}, t_0) \geq M(x_{m_k+1}, x_{m_k}, t_0) * M(x_{m_k}, x_{n_k}, t_0) * M(x_{n_k}, x_{n_k+1}, t_0)$$

and

$$M(x_{m_k}, x_{n_k}, t_0) \geq M(x_{m_k}, x_{m_k+1}, t_0) * M(x_{m_k+1}, x_{n_k+1}, t_0) * M(x_{n_k+1}, x_{n_k}, t_0),$$

imply that

$$\lim_{k \rightarrow +\infty} M(x_{m_k+1}, x_{n_k+1}, t_0) = 1 - \varepsilon.$$

From (28), we have

$$M(gx_{m_k+1}, Tx_{m_k}, t_0) = M(A, B, t_0) \quad \text{and} \quad M(gx_{n_k+1}, Tx_{n_k}, t_0) = M(A, B, t_0).$$

Thus

$$M(x_{m_k}, x_{n_k}, t_0) \leq \eta[M(gx_{m_k+1}, gx_{n_k+1}, t_0)] \leq \eta[M(x_{m_k+1}, x_{n_k+1}, t_0)] < M(x_{m_k+1}, x_{n_k+1}, t_0).$$

On taking the limit as $k \rightarrow \infty$ in the above inequality, we get $1 - \varepsilon \leq \eta(1 - \varepsilon) < 1 - \varepsilon$, a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in the closed subset $A(t)$ of complete partially ordered fuzzy metric space $(X, M, *, \leq)$. Thus there exists $x^* \in A(t)$ such that $\lim_{n \rightarrow \infty} M(x_n, x^*, t) = 1$, for all $t > 0$. This further implies that $M(gx^*, Tx^*, t) = \lim_{n \rightarrow \infty} M(gx_{n+1}, Tx_n, t) = M(A, B, t)$ and hence $x^* \in A_0(t)$ is the optimal coincidence point of a pair $\{g, T\}$. To prove the uniqueness of x^* , we show that, for any fixed element $\bar{x}_0 \in A_0(t)$, the sequence $\{\bar{x}_n\} \in A_0(t)$ defined by $M(g\bar{x}_{n+1}, T\bar{x}_n, t) = M(A, B, t)$ converges to x^* . Suppose that there is another element $\bar{x}_0 \in A(t)$ such that $0 < M(x_0, \bar{x}_0, t) < 1$ for all $t > 0$ satisfying

$$M(g\bar{x}_0, T\bar{x}_0, t) = M(A, B, t). \quad (34)$$

Suppose that $(\bar{x}_0, x_0) \in \Delta_{(t)}$. Then, by the given assumption, we have

$$M(\bar{x}_0, x_0, t) \leq \eta(M(g\bar{x}_0, gx_0, t)) \leq \eta(M(\bar{x}_0, x_0, t)) < M(\bar{x}_0, x_0, t),$$

a contradiction and hence the result follows. If $(\bar{x}_0, x_0) \notin \Delta_{(t)}$, then let u_0 be a lower bound of x_0 and \bar{x}_0 , and \bar{u}_0 an upper bound of x_0 and \bar{x}_0 . That is,

$$\bar{u}_0 \geq x_0 \geq u_0 \quad \text{or} \quad \bar{u}_0 \geq \bar{x}_0 \geq u_0.$$

Recursively, construct sequences $\{u_n\}$ and $\{\bar{u}_n\}$, such that

$$M(gu_{n+1}, Tu_n, t) = M(A, B, t) \quad \text{and} \quad M(g\bar{u}_{n+1}, T\bar{u}_n, t) = M(A, B, t).$$

The proximal monotonicity of the mapping T and the monotonicity of the inverse of g imply that

$$\bar{u}_n \geq \bar{x}_n \geq u_n \quad \text{or} \quad \bar{u}_n \leq \bar{x}_n \leq u_n.$$

From $(x_n, u_n) \in \Delta_{(t)}$ and $(x_n, \bar{u}_n) \in \Delta_{(t)}$, it follows that

$$\lim_{n \rightarrow \infty} \bar{u}_n = \lim_{n \rightarrow \infty} u_n = x^*.$$

Hence $\lim_{n \rightarrow \infty} \bar{x}_n = x^*$. □

Example 4.2 Let $X = [-1, 1] \times \mathbb{R}$ and \leq a usual order on \mathbb{R}^2 . Let $A = \{(-1, x) : \text{for all } x \in \mathbb{R}\}$, $B = \{(1, y) : \text{for all } y \in \mathbb{R}\}$, and $(X, M, *, \leq)$ a complete fuzzy ordered metric space as given in Example 2.17. Note that $M(A, B, t) = \frac{t}{t+2}$, $A_0(t) = A$ and $B_0(t) = B$. Define $T : A \rightarrow B$ by

$$T(-1, x) = \left(1, \frac{x}{5}\right).$$

Let $g : A \rightarrow A$ be defined by $g(-1, x) = (-1, \frac{x}{2})$. Note that g is fuzzy nonexpansive and its inverse is monotone. Obviously, $T(A_0(t)) \subseteq B_0(t)$, and $A_0(t) \subseteq g(A_0(t))$. Note that $u = (-1, \frac{2}{5}y_1)$, $v = (-1, \frac{2}{5}y_2)$, $x = (-1, y_1)$, and $y = (-1, y_2) \in A$. Also, note that

$$M((-1, y_1), (-1, y_2), t) \leq \eta \left(M \left(\left(-1, \frac{y_1}{5}\right), \left(-1, \frac{y_2}{5}\right), t \right) \right).$$

Here $\eta(t) = 2t - t^2$. Thus all conditions of Theorem 4.1 are satisfied. Moreover, $(-1, 0)$ is the optimal coincidence point of g and T .

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics and Applied Mathematics, University of Pretoria, Hatfield, Pretoria, South Africa.

²Department of Mathematics, National University of Computer and Emerging Sciences, Lahore, Pakistan. ³Institute of Research and Development of Processes, University of the Basque Country, Campus of Leioa (Bizkaia), Barrio Sarriena, P.O. Box 644, Bilbao, 48940, Spain.

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